The Schulze Method

Markus Schulze proposed a voting rule that is sometimes known as beatpath, sometimes as cloneproof Schwartz sequential dropping, and sometimes as variations on these names.[1] Since there are two quite different ways to determine the winner, Schulze suggests that it be called the Schulze method, so that the name will not appear to exclude either of these ways.[2]

The "beatpath" way to determine the winner of the Schulze method employs a transitive relation among options, defined as follows. A "path" from option x to option y is a sequence of options that begins with x and ends with y. A path is a "chain" if, for every pair, (i, j), of successive options in the path, $M(i, j) > 0$. Just as a chain is only as strong as its weakest link, the "strength" of a chain is the minimum of the majorities of the pairs of successive options in the chain.

If majority rule yields a transitive ordering of the options, then for every pair of options (x, y) for which $M(x, y) > 0$, there is a chain from x to y and there is no chain from y to x. But if x and y are in a cycle, then there is both a chain from x to y and a chain from y to x. If the cycle involves multiple options, then there can be many chains from x to y and many from y to x.

For any ordered pair of options (x, y), define $S(x, y)$ to be the strength of the strongest chain from x to y if there are any chains from x to y, and 0 if there are no chains from x to y. Define a relation between pairs of options, D (disqualifies), by $x \; D \; y$ if and only if $S(x, y) > S(y, x)$. The proof that follows shows that D is transitive.[3]

Suppose that $x \; D \; y$ and $y \; D \; z$. Then:

$$S(x, y) > S(y, x); \tag{13.3}$$

$$S(y, z) > S(z, y). \tag{13.4}$$

Since one path from x to z is the strongest chain from x to y followed by the strongest chain from y to z,

$$S(x, z) \geq \text{Min}[S(x, y), S(y, z)]. \tag{13.5}$$

Applying this idea to other pairs of links,

$$S(y, x) \geq \text{Min}[S(y, z), S(z, x)]; \tag{13.6}$$

$$S(z, y) \geq \text{Min}[S(z, x), S(x, y)]; \tag{13.7}$$

Consider separately the cases of $S(x, y) \geq S(y, z)$ and $S(x, y) < S(y, z)$. If

$$S(x, y) \geq S(y, z), \tag{13.8}$$

then in view of (13.4),

$$S(x, y) > S(z, y). \tag{13.9}$$

This says that one of the two items on the right side of (13.7) is greater than the left side of (13.7), so it must be the other item on the right side of (13.7) that is the smaller, so that

$$S(z, y) \geq S(z, x). \tag{13.10}$$

From (13.8), the second of the two items on the right side of (13.5) is the smaller, so that

$$S(x, z) \geq S(y, z). \tag{13.11}$$

Combining (13.11), (13.4) and (13.10),

$$S(x, z) \geq S(y, z) > S(z, y) \geq S(z, x), \tag{13.12}$$

which means that $x \, D \, z$.

Now consider the case of $S(x, y) < S(y, z)$. Reversing the two sides,

$$S(y, z) > S(x, y), \tag{13.13}$$

which, in view of (13.3), implies that

$$S(y, z) > S(y, x). \tag{13.14}$$

This says that one of the two items on the right side of (13.6) is greater than the left side of (13.6), so it must be the other item on the right side of (13.6) that is the smaller, so that

$$S(y, x) \geq S(z, x). \tag{13.15}$$

From (13.13), the second of the two items on the right side of (13.5) is the greater, so that

$$S(x, z) \geq S(x, y). \tag{13.16}$$

Combining (13.16), (13.3), and (13.15),

$$S(x, z) \geq S(x, y) > S(y, x) \geq S(z, x), \tag{13.17}$$

which again means that $x\ D\ z$, so that D is transitive.

The "beatpath" definition of the Schulze method is that the winning options in an election are the options that are not disqualified by any other option by the relation D. Since D is transitive, there is at least one winning option. If there is more than one winning option, the tie among the winning options is broken by a TBRO, constructed in the way described in the discussion of the ranked pairs rule.

One way to determine the beatpath winner is to calculate the strength of every path from every option to every other option and determine which options are not disqualified. Schulze offers an efficient algorithm for doing this.[4] But it is not necessary to use such an algorithm to determine the winner under the Schulze method.

When paired comparisons by majority rule establish a transitive order of the options, the Schulze method selects the dominant option, since every option in such a transitive order disqualifies every option below it. (No chain can go up a transitive order.) When the Smith set for an election contains more than one option, the Schulze method selects something from the Smith set, since every option inside the Smith set disqualifies every option outside it. (No chain can enter the Smith set from outside.)

In fact, if the GOCHA set is smaller than the Smith set, then the winner by the Schulze method must come from the GOCHA set. The reason is as follows. For some election, define G as the GOCHA set, S as the set of options that are in the Smith set but not in G, and E as the set of all other options (which may be the empty set). By the definition of the Smith set, all options in G and S beat all options in E. If all the options in G beat all the options in S, then the options in S would be in E. If any of the options in S beat any of the options in G, then at least some of the options in S would be in G. If none of the options in G beat any of the options in S, then the options in S would be in G. This means that if S is not empty, then in paired comparisons between an element of G and an element of S, there must be some cases in which the element of G wins, some in which there is a tie between the element of G and the element of S, and none in which the element of S wins. Therefore there is at least one chain that goes from an element of G to and element of S and no chain that goes from any element of S to any element of G. Therefore any s in S to which a chain from an element of G goes is disqualified, as are all elements of S that are beaten by s, and all that are beaten by these, and so on until no more are added. If there were elements of S that could not be disqualified in this way, then they would be an undominated set, and any minimal undominated subset of them would be in G and not in S. Thus all elements of S are disqualified, implying that the winner under the Schulze method must be an element of the GOCHA set. This means that it is generally easiest to determine the winner under the Schulze method by first determining the GOCHA set and then determining which elements of the GOCHA set are disqualified.

One can finish the task of identifying the winner under the Schulze method by determining the strengths of all of the chains within the GOCHA set and thereby

determine which options are disqualified, but there is a method of successive eliminations that is simpler. Let m_1 be the minimum of the majorities between pairs of options in the GOCHA set. If some option, x, has a chain to some other option, y, of strength m_1, then x will be disqualified by y unless it happens that the chain from y to x also has strength m_1. The disqualifications will be the same as if every majority of m_1 is replaced by a majority of 0. As long as some majorities in the GOCHA set are greater than m_1, one can replace all majorities of m_1 with majorities of 0 and identify a new GOCHA set in which the winning option must lie. Any options that were in the previous GOCHA set and not in the new GOCHA set are disqualified.

Let m_2 be the minimum of the majorities between pairs of options in the new GOCHA set. As long as some majorities in the new GOCHA set are greater than m_2, one can replace all majorities of m_2 with majorities of 0 and identify a new GOCHA set in which the winning option must lie. This process continues until all majorities in the GOCHA set are equal. All options then in the GOCHA set are the winners. These are the options that do not disqualify each other and do disqualify all options not in the final GOCHA set. Thus they are the same as the winners under the beatpath description of the Schulze method.

The Schulze method possesses Schwartz consistency, as shown above (and therefore Smith consistency, Condorcet consistency and majority consistency), mutual majority consistency, invariant loss consistency, monotonicity, homogeneity, and independence of clones. It lacks positive responsiveness, perfect-tie responsiveness, and combinativity. I do not know whether the Schulze method is resolvable.

The Schulze method possesses mutual majority consistency because if there is a set of options, S, such that a majority of voters rank all options in S ahead of all other options, then no option outside of S could be in the Smith set. Since the Schulze method possesses Smith consistency, it selects a winner from S, so it possesses mutual majority consistency. Similarly, The Schulze method possesses invariant loss consistency because an option that loses to all other options cannot be in the Smith set.

The Schulze method possesses monotonicity by the following argument. Suppose that the sequence of rankings $\mathbf{R}'$ is an incremental upward movement for x relative to the sequence $\mathbf{R}$, and that x is among the winning options for $\mathbf{R}$. For any chain from x to an option that is disqualified by x, the incremental upward movement either increases the strength of the chain or has no effect on it, while for any chain to x, the incremental upward movement either reduces the strength of the chain or leaves it unaffected. Thus under rankings $\mathbf{R}'$, x continues to disqualify all options that it disqualified under $\mathbf{R}$. For any options that were not disqualified by x under $\mathbf{R}$ (which would be options that were tied with x for winning status), the incremental upward movement might cause x to now disqualify them and could not cause them to disqualify x. Thus x continues to be a winning option.

The Schulze method possesses homogeneity because replicating a sequence of rankings n times causes all $M(i, j)$ to be multiplied by n and has no effect on the pattern of disqualifications.

The Schulze method is independent of clones by the following argument. If *c*', a clone of *c*, is added, then for every *x* that is not a clone of *c*, $M(c', x) = M(c, x)$, so that *c*' disqualifies *x* if and only if *c* disqualifies *x*, and *x* disqualifies *c*' if and only if *x* disqualifies *c*. Therefore if a clone is a winner before *c*' is added, a clone remains among the winner after *c*' is added, and any *x* that is a winner before *c*' is added remains a winner after *c*' is added.

Example 13.6 shows that the Schulze method lacks positive responsiveness. By the Schulze method, the election is a tie between *x* and *y* both before and after the incremental upward movement for *x*.

Example 13.7 shows that the Schulze method lacks perfect-tie responsiveness. The addition to the perfect tie of a vote ranking *x* first leaves a tie between *x* and *z* by the Schulze method.

Example 13.8 shows that the Schulze method lacks combinativity. It awards the elections for sequences **R** and **R**′ to *x* and the election for sequence **R** + **R**′ to *z*.

Turning to measures of qualitative attractiveness, I give the Schulze method a score of 4 for lucidity, a notch below the Young rule and the ranked pairs rule. I give it a score of 5 for ease of use, like all rules that require rankings. I give it a score of 4 for cost of computation, a notch below the ranked pairs rule, because of the relative difficulty of identifying the GOCHA set.

The opportunities for strategy under the Schulze method are likely to be the same as for the Condorcet, simplified Dodgson, Nanson and Maximin rules when truthful votes do not generate cycles, so its score for resistance to strategy is 8.9.

The Schulze method is substantially similar to the ranked pars rule, but definitely different since it possesses Schwartz consistency and the ranked pairs rule does not. A choice between the two could reasonably be based on whether one believes that a vote processing rule ought to possess Schwartz consistency.

Notes

1 M. Schulze, "A New Monotonic and Clone-Independent Single-Winner Election Method," *Voting Matters* 17 (2003): 9–19 (www.mcdougall.org.uk/VM/MAIN.HTM).
2 Ibid., p.9
3 The proof is based on one by Schulze in Ibid., pp. 10–11.
4 Ibid., p. 11.